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THE MODULI SPACES OF PARABOLIC CONNECTIONS WITH A QUADRATIC DIFFERENTIAL AND ISOMONODROMIC DEFORMATIONS

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1. Introduction

Let (C, \mathbf{t}) ($\mathbf{t} = (t_1, \dots, t_n)$) be an n -pointed smooth projective curve of genus g over \mathbb{C} where t_1, \dots, t_n are distinct points. We take a positive integer r , and take an element $\nu = (\nu_j^{(i)})_{\substack{1 \leq i \leq n \\ 0 \leq j \leq r-1}} \in \mathbb{C}^{nr}$ such that $\sum_{i,j} \nu_j^{(i)} = -d \in \mathbb{Z}$. Put $D = t_1 + \dots + t_n$.

Definition of parabolic connections

We say (E, ∇, \mathbf{I}) ($\mathbf{I} = \{I_*^{(i)}\}_{1 \leq i \leq n}$) is a ν -parabolic connection of rank r on (C, \mathbf{t}) if

- E is a rank r algebraic vector bundle of degree d on C ,
- $\nabla: E \rightarrow E \otimes \Omega_C^1(t_1 + \dots + t_n)$ is a connection, and
- for each t_i , $I_*^{(i)}$ is a filtration $E|_{t_i} = I_0^{(i)} \supset I_1^{(i)} \supset \dots \supset I_r^{(i)} = 0$ such that $\dim(I_j^{(i)}/I_{j+1}^{(i)}) = 1$ and $(\text{res}_{t_i}(\nabla) - \nu_j^{(i)} \text{id}_{E|_{t_i}})(I_j^{(i)}) \subset I_{j+1}^{(i)}$ for $j = 0, \dots, r-1$.

Take rational numbers

$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \dots < \alpha_r^{(i)} < 1$$

for $i = 1, \dots, n$ satisfying $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i, j) \neq (i', j')$. We choose a sufficiently generic $\alpha = (\alpha_j^{(i)})$.

α -stability of parabolic connections

A parabolic connection (E, ∇, \mathbf{I}) is α -stable if for any proper nonzero subbundle $F \subset E$ satisfying $\nabla(F) \subset F \otimes \Omega_C^1(t_1 + \dots + t_n)$, the inequality

$$\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim(F|_{t_i} \cap I_{j-1}^{(i)}) / (F|_{t_i} \cap I_j^{(i)})}{\text{rank } F} < \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim(I_{j-1}^{(i)} / I_j^{(i)})}{\text{rank } E}$$

holds.

Moduli space of α -stable ν -parabolic connections [2]

$$M^\alpha(r, d, \nu) = \left\{ ((C, \mathbf{t}), (E, \nabla, \mathbf{I})) \mid \begin{array}{l} (E, \nabla, \mathbf{I}) \text{ is } \alpha\text{-stable} \\ \nu\text{-parabolic connection on } (C, \mathbf{t}) \end{array} \right\} / \sim$$

Moduli space of α -stable ν -parabolic connections with a quadratic differential

$$\begin{array}{ccc} \widehat{M}^\alpha(r, d, \nu) & \rightarrow & M^\alpha(r, d, \nu) \\ \pi \downarrow & & \downarrow \pi \\ T^* \widehat{M}_{g,n} & \rightarrow & \widehat{M}_{g,n}. \end{array}$$

where $\widehat{M}_{g,n}$ is a smooth algebraic scheme which is a certain covering of the moduli stack of n -pointed smooth projective curves of genus g over \mathbb{C} .

2. Deformation theory of parabolic connections with a quadratic differential

$((C, \mathbf{t}, \psi), (E, \nabla, \mathbf{I})) \in \widehat{M}^\alpha(r, d, \nu)$, where $\psi \in H^0(C, \Omega_C^2(D))$.

Definition of \mathcal{F}_C^0 and \mathcal{F}_C^1

$$\tilde{\mathcal{F}}_C^0 := \left\{ s \in \mathcal{E}nd(E) \mid s|_{t_i}(I_j^{(i)}) \subset I_j^{(i)} \text{ for any } i, j \right\}$$

$$0 \longrightarrow \tilde{\mathcal{F}}_C^0 \longrightarrow \mathcal{F}_C^0 \xrightarrow{\text{symp}_1} \Theta_C(-D) \longrightarrow 0$$

$$\tilde{\mathcal{F}}_C^1 := \left\{ s \in \mathcal{E}nd(E) \otimes \Omega_C^1(D) \mid \text{res}_{t_i}(s)(I_j^{(i)}) \subset I_{j+1}^{(i)} \text{ for any } i, j \right\};$$

$$\mathcal{F}_C^1 := \tilde{\mathcal{F}}_C^1 \oplus \Omega_C^2(D)$$

Transformation of elements of \mathcal{F}_C^0

$$a_i \frac{\partial}{\partial z_i} + \eta_i \longmapsto a_i \frac{\partial z_j}{\partial z_i} \frac{\partial}{\partial z_j} + \theta_{ij}^{-1} \eta_i \theta_{ij} + \theta_{ij}^{-1} \frac{\partial \theta_{ij}}{\partial z_i},$$

Section of symp_1 for $\nabla (= d + A_i dz_i)$

$$\begin{aligned} \iota(\nabla): \Theta_C(-D) &\longrightarrow \mathcal{F}_C^0 \\ a_i \frac{\partial}{\partial z_i} &\longmapsto a_i \frac{\partial}{\partial z_i} + a_i A_i \end{aligned}$$

The complex \mathcal{F}_C^\bullet

$$d_{\mathcal{F}_C^\bullet} := (d_\nabla, d_\psi) \circ (\text{Id} - \iota(\nabla) \circ \text{symp}_1, \text{symp}_1): \mathcal{F}_C^0 \longrightarrow \mathcal{F}_C^1,$$

where $d_\nabla: \tilde{\mathcal{F}}_C^0 \rightarrow \tilde{\mathcal{F}}_C^1$; $s \mapsto \nabla \circ s - s \circ \nabla$ and $d_\psi: \Theta_C(-D) \rightarrow \Omega_C^2(D)$ defined by

$$a_i \frac{\partial}{\partial z_i} \longmapsto \left(\frac{\partial \psi_i}{\partial z_i} a_i + 2\psi_i \frac{\partial a_i}{\partial z_i} \right) dz_i^{\otimes 2}$$

$H^1(C, \mathcal{F}_C^\bullet)$ is the tangent space of $\widehat{M}^\alpha(r, d, \nu)$ at $((C, \mathbf{t}, \psi), (E, \nabla, \mathbf{I}))$.

Symplectic structure

For $v = [(\{u_{ij}\}, \{(v_i, w_i)\})] \in H^1(C, \mathcal{F}_C^\bullet)$ and $w = [(\{u'_{ij}\}, \{(w'_i, w'_i)\})] \in H^1(C, \mathcal{F}_C^\bullet)$, we put

$$\begin{aligned} \omega_1(v, w) &= [(\{\text{Tr}(\eta(u_{ij}) \circ \eta(u'_{jk}))\}, -\{\text{Tr}(\eta(u_{ij}) \circ v'_j) - \text{Tr}(v_i \circ \eta(u'_{ij}))\})] \text{ and} \\ \omega_2(v, w) &= [2 \cdot \text{symp}_1(u_{ij}) \circ \text{symp}_1(u'_{jk}) \circ \psi_j, -\{\text{symp}_1(u_{ji}) \circ w'_j + w_i \circ \text{symp}_1(u'_{ij})\}]. \end{aligned}$$

where $\eta(s) := s - \iota(\nabla) \circ \text{symp}_1(s)$.

$$\omega := \omega_1 + \omega_2: H^1(C, \mathcal{F}_C^\bullet) \otimes H^1(C, \mathcal{F}_C^\bullet) \longrightarrow H^2(C, \Omega_C^\bullet) \cong \mathbb{C}$$

3. Cotangent bundle on moduli space of curves with a bundle

Let $\mathfrak{P}_{g,n}(r, d)$ be the moduli stack of pairs $((C, \mathbf{t}), (E, \mathbf{I}))$, where (C, \mathbf{t}) ($\mathbf{t} = (t_1, \dots, t_n)$) is an n -pointed smooth projective curve of genus g over \mathbb{C} where t_1, \dots, t_n are distinct points, and (E, \mathbf{I}) is a quasi-parabolic bundle of rank r and of degree d on (C, \mathbf{t}) .

Let $\mathfrak{P}_{g,n}^\alpha(r, d, \nu)$ be the substack defined by the condition where a quasi-parabolic bundle admits an α -stable ν -parabolic connection.

Tangent space at $((C, \mathbf{t}), (E, \mathbf{I}))$

$H^1(C, \mathcal{F}_C^1)$ is the tangent space of $\mathfrak{P}_{g,n}^\alpha(r, d, \nu)$ at $((C, \mathbf{t}), (E, \mathbf{I}))$.

Cotangent space at $((C, \mathbf{t}), (E, \mathbf{I}))$

$$0 \longrightarrow \Omega_C^2(D) \longrightarrow \mathcal{H}_C^1 \longrightarrow (\tilde{\mathcal{F}}_C^0)^* \otimes \Omega_C^1 \longrightarrow 0.$$

Note that $(\tilde{\mathcal{F}}_C^0)^* \otimes \Omega_C^1 \cong \tilde{\mathcal{F}}_C^1$.

$H^0(C, \mathcal{H}_C^1)$ is the cotangent space of $\mathfrak{P}_{g,n}^\alpha(r, d, \nu)$ at $((C, \mathbf{t}), (E, \mathbf{I}))$.

Transformation of elements of \mathcal{H}_C^1

$$\begin{aligned} (\Phi_i(z_i) dz_i, \phi_i(z_i) dz_i^{\otimes 2}) &\longmapsto (\Phi_j(z_i) dz_i, \phi_j(z_i) dz_i^{\otimes 2}) \\ &:= \left((\theta_{ij}^{-1} \Phi_i(z_i) \theta_{ij}) dz_i, \phi_i(z_i) d\theta_{ij}^{\otimes 2} + \text{Tr} \left(\theta_{ij}^{-1} \Phi_i(z_i) \frac{\partial \theta_{ij}}{\partial z_i} \right) dz_i^{\otimes 2} \right), \end{aligned}$$

Extended parabolic Higgs bundles

We call $((C, \mathbf{t}), (E, \mathbf{I}), \widehat{\Phi})$, where $\widehat{\Phi} \in H^0(C, \mathcal{H}_C^1)$, an *extended parabolic Higgs bundle*. We call $\widehat{\Phi} \in H^0(C, \mathcal{H}_C^1)$ an *extended parabolic Higgs field*.

The complex \mathcal{H}_C^\bullet

For an extended parabolic Higgs bundle $((C, \mathbf{t}), (E, \mathbf{I}), \widehat{\Phi})$ ($\widehat{\Phi} = (\Phi_i dz_i, \phi_i dz_i^{\otimes 2})$ on U_i),

$$\begin{aligned} d_{\mathcal{H}_C^\bullet}: \mathcal{F}_C^0 &\longrightarrow \mathcal{H}_C^1 \\ a_i \frac{\partial}{\partial z_i} + \eta_i &\longmapsto \left(\Phi_i dz_i \circ \eta_i - \eta_i \circ \Phi_i dz_i - \frac{\partial(a_i \Phi_i)}{\partial z_i} dz_i, \right. \\ &\quad \left. \text{Tr} \left(\frac{\partial \eta_i}{\partial z_i} \Phi_i dz_i^{\otimes 2} \right) - a_i \frac{\partial \phi_i}{\partial z_i} dz_i^{\otimes 2} - 2 \frac{\partial a_i}{\partial z_i} \phi_i dz_i^{\otimes 2} \right). \end{aligned}$$

$H^1(C, \mathcal{H}_C^\bullet)$ is the space of first order deformations of $((C, \mathbf{t}), (E, \mathbf{I}), \widehat{\Phi})$.

Symplectic structure

$$\omega_H: H^1(C, \mathcal{H}_C^\bullet) \otimes H^1(C, \mathcal{H}_C^\bullet) \longrightarrow H^2(C, \Omega_C^\bullet) \cong \mathbb{C}$$

which is defined by

$$\begin{aligned} &[(\{a_{ij} \partial / \partial f_i + \eta_{ij}\}, \{(\tilde{v}_i, \tilde{w}_i)\})] \otimes [(\{a'_{ij} \partial / \partial f_i + \eta'_{ij}\}, \{(\tilde{v}'_i, \tilde{w}'_i)\})] \\ &\longmapsto [(\{ \text{Tr}(\eta_{ji}(a'_{jk} \Phi_j)) + \text{Tr}((a_{ij} \Phi_i) \eta'_{jk}) - 2a_{ji} a'_{jk} \phi_j, \\ &\quad - \{ -\text{Tr}(\eta_{ji} \tilde{v}'_j) + (a_{ji} \tilde{w}'_j) - \text{Tr}(\tilde{v}_i \eta_{ij}) + (\tilde{w}_i a'_{ij}) \} \})]. \end{aligned}$$

5. Twisted cotangent bundle on moduli space of curves with a bundle

Definition of twisted cotangent bundles [1]

Let $T^* = T^*(X) \rightarrow X$ be the cotangent bundle on X . A *twisted cotangent bundle* on X is a T^* -torsor $\pi_\phi: \phi \rightarrow X$ (i.e., π_ϕ is a fibration equipped with a simple transitive action of T^* along the fibers) together with a symplectic form ω_ϕ on ϕ such that π_ϕ is a polarization for ω_ϕ (i.e., $\dim \phi = 2 \dim X$ and the Poisson bracket $\{\cdot, \cdot\}$ vanished on $\pi_\phi^{-1} \mathcal{O}_X$) and for any 1-form ν one has $t_\nu^*(\omega_\phi) = \pi_\phi^* d\nu + \omega$. Here $t_\nu: \phi \rightarrow \phi$; $t_\nu(a) = a + \nu_{\pi(a)}$ is the translation by ν .

Let T^* be the moduli stack of pairs $((C, \mathbf{t}), (E, \mathbf{I}), \widehat{\Phi})$, where $(C, \mathbf{t}), (E, \mathbf{I}) \in \mathfrak{P}_{g,n}^\alpha(r, d, \nu)$ and $\widehat{\Phi}$ is an extended parabolic Higgs field on $((C, \mathbf{t}), (E, \mathbf{I}))$:

$$T^* \longrightarrow \mathfrak{P}_{g,n}^\alpha(r, d, \nu).$$

Let $\widehat{\mathfrak{M}}^\alpha(r, d, \nu)$ be the moduli stack corresponding to $\widehat{M}^\alpha(r, d, \nu)$:

$$\pi_{\widehat{\mathfrak{M}}^\alpha(r, d, \nu)}: \widehat{\mathfrak{M}}^\alpha(r, d, \nu) \longrightarrow \mathfrak{P}_{g,n}^\alpha(r, d, \nu).$$

T^* -torsor structure on $\pi_{\widehat{\mathfrak{M}}^\alpha(r, d, \nu)}$

Action of an extended parabolic Higgs field $\widehat{\Phi} = (\Phi_i dz_i, \phi_i dz_i^{\otimes 2})$ on a parabolic connection with quadratic differential $(d + A_i dz_i, \psi_i dz_i^{\otimes 2})$:

$$(d + A_i dz_i, \psi_i dz_i^{\otimes 2}) \longmapsto \left(d + (A_i + \Phi_i) dz_i, \psi_i dz_i^{\otimes 2} + \phi_i dz_i^{\otimes 2} - \text{Tr} \left(\Phi_i A_i + \frac{1}{2} \Phi_i \Phi_i \right) dz_i^{\otimes 2} \right),$$

Main result

$(\pi_{\widehat{\mathfrak{M}}^\alpha(r, d, \nu)}, \omega)$ is a twisted cotangent bundle on $\mathfrak{P}_{g,n}^\alpha(r, d, \nu)$.

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4. Deformation theory of extended parabolic Higgs bundles